The Degasperis–Procesi equation and its short-wave limit by a Riemann–Hilbert approach

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Degasperis–Procesi and Ostrovsky–Vakhnenko equations

- Degasperis–Procesi (DP):

$$u_t - u_{txx} + 3\varkappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}$$

- Ostrovsky–Vakhnenko (OV):

$$u_{txx} - 3\varkappa u_x + 3u_x u_{xx} + uu_{xxx} = 0$$

Cauchy problem:

$$u(x,0) = u_0(x), \qquad x \in (-\infty, \infty),$$

where $u_0(x)$ is smooth and decays sufficiently fast as $|x| \to \infty$, and

- $u_0(x) - u_{0xx}(x) + \varkappa > 0$ for all x
- $-u_{0xx}(x) + \varkappa > 0$

Then $u(x,t)$ exists globally and $u_x(x,t) - u_{xx}(x,t) + \varkappa > 0$ $(-u_{xx}(x,t) + \varkappa > 0)$ for all (x,t)).
In what follows: $\varkappa > 0$.

Shallow water wave models

- result from approximations to “full” equations (Euler, Green–Naghdi) governing the motion of inviscid fluid whose surface can exhibit gravity wave propagation
- small parameters: $\varepsilon = \dfrac{a}{h}$, $\mu = \dfrac{h^2}{\lambda^2}$, where
  - a is the typical amplitude
  - h is the mean depth
  - λ is the typical wavelength

▷ shallow water scaling: $\mu << 1$ weakly dispersive

▷ long-wave: $\varepsilon << 1$ weakly nonlinear

Integrable models in the shallow water theory

- long-wave regime: $\mu << 1$, $\varepsilon = \mathrm{O}(\mu)$; balance between nonlinearity and dispersion

$$u_t + u_x + uu_x + u_{xxx} = 0 \quad \text{(KdV)}$$

- "Camassa–Holm scaling": $\mu << 1$, $\varepsilon = \mathrm{O}(\sqrt{\mu})$; more nonlinear than dispersive; can allow breaking waves. From "b-family"

$$u_t - u_{txx} + \varkappa u_x + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}$$

  integrable (possessing the Lax pair) are:
  - $b = 2$: Camassa–Holm equation (1993) (also Fokas, Fuchssteiner, 1981)
  - $b = 3$: Degasperis–Procesi equation (1999)

- Camassa-Holm equation (CH)

$$u_t - u_{txx} + 2\varkappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

  - $\varkappa > 0$ is related to the critical wave speed $\sqrt{gh_0}$; supports smooth solitary waves (like for KdV)
  - $\varkappa = 0$: supports peakons $c \cdot e^{-|x-ct|}$
    - waves in hyperelastic rods (Dai, 1998)
    - geodesic flow on diffeomorphism group of the line (Misiolek, 1998)

- Degasperis–Procesi equation (DP)

$$u_t - u_{txx} + 3\varkappa u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}$$

Solutions on zero background (spatially decaying):

- $\varkappa > 0$ smooth solitary waves
- $\varkappa = 0$: peakons

Short wave limit: $\varepsilon \to 0$

$$x' = \frac{x}{\varepsilon}, \quad t' = t\varepsilon, \quad u' = \frac{u}{\varepsilon^2}$$

- CH ($\varkappa = 0$) $\longmapsto$ $(u_t + uu_x)_{xx} = u_x u_{xx}$
  Hunter-Saxton equation: nematic liquid crystals
- CH ($\varkappa > 0$) $\longmapsto$ $u_{txx} - 2\varkappa u_x + 2u_x u_{xx} + uu_{xxx} = 0$
  "modified Hunter-Saxton equation": short capillary waves
- DP ($\varkappa = 0$) $\longmapsto$ $(u_t + uu_x)_{xx} = 0$
  derivative Burgers equation
- DP ($\varkappa > 0$) $\longmapsto$ $u_{txx} - 3\varkappa u_x + 3u_x u_{xx} + uu_{xxx} = 0$
  Ostrovsky–Vakhnenko equation

OV equation: various physics and many names

- Whitham (1974) $u_t + uu_x + \int_{-\infty}^{\infty} K(x-y)u_y dy = 0$ (wave equations with breaking & peaking), with $K(x-y) = \frac{1}{2}|x-y|$
- "Vakhnenko equation" (1991): high-frequency waves in a relaxing medium. $(u_t + uu_x)_x + u = 0$. Name proposed by J.Parkes, 1993.
- "reduced Ostrovsky equation" (1978): weakly nonlinear surface and internal waves in a rotating ocean influenced by Earth rotation. $(u_t + c_0 u_x + \alpha u u_x + \beta u_{xxx})_x = \gamma u$ ($\beta = 0$)
- "Rotation-Modified KdV equation" (RMKdV); "Ostrovsky–Hunter" equation (Hunter (1990): canonical asymptotic equation for genuinely nonlinear waves that are non-dispersive as their wavelength tends to zero).

Short wave (sw-) limits: integrable as well as their counterparts (possess Lax pair representation).

- CH and sw-CH: spatial equation in the Lax pair is of second order
- DP and sw-DP (OV): spatial equation in the Lax pair is of third order

Accordingly, in the application of the inverse scattering method, in the form of the matrix-valued Riemann–Hilbert problem,

- CH and sw-CH: 2×2 Riemann–Hilbert problem

  CH: [Constantin, 2001], [Constantin,Lenells, 2003], [Boutet de Monvel, Sh., 2006,...], [Constantin, Gerdjikov, Ivanov, 2006], [Boutet de Monvel, Kostenko, Sh., Teschl, 2010]

- DP and OV: 3×3 Riemann–Hilbert problem

  DP: [Constantin, Ivanov, Lenells, 2010], [Boutet de Monvel, Sh., 2012]

Objective: development of the RHP method for the initial-value problem, in view of studying the long-time asymptotics.

Implementation in the case of short wave limits:

- share some common features with “original” equations
- have specific features, both in the realization and in the asymptotic results

In this talk: DP versus OV

DP ($\varkappa = 1$) on $(-\infty, \infty)$ with fast decaying initial data

Let $u(x,t)$ be the solution of the initial-value problem for the Degasperis-Procesi equation (for $\varkappa = 1$):

- $u_t - u_{txx} + 3u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}$
- $u(x,0) = u_0(x)$

Let $m(x,t) := u(x,t) - u_{xx}(x,t)$.

Assumptions:

- $u_0(x) \to 0$ as $|x| \to \infty$
- $m(x,0) + 1 > 0$ for all $x \in \mathbb{R}$ (then $m(x,t) + 1 > 0$ for all x, t)

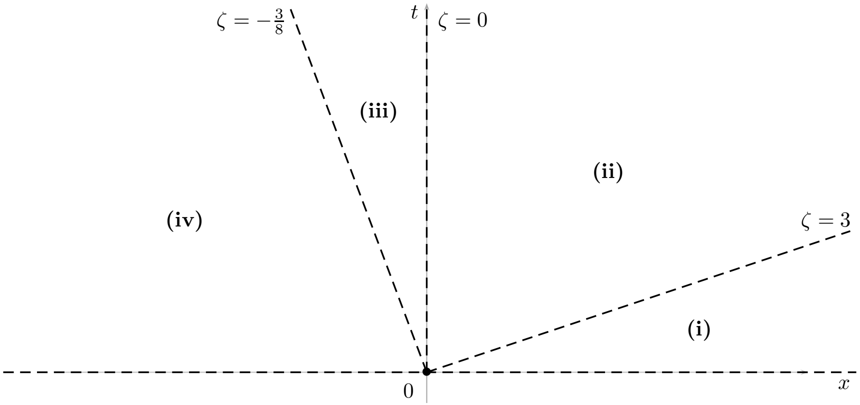
Question

How does $u(x,t)$ behave for large t?

Answer

- four sectors in the (x,t) half-plane where $u(x,t)$ behaves differently for large t, depending on the magnitude of $\zeta = \frac{x}{t}$
- transition zones (Painlevé)

DP: long-time asymptotics



Four sectors and transition zones in the (x, t)-half-plane, $\zeta = \frac{x}{t}$.
Painlevé zones: (a) $|\zeta - 3|t^{2/3} < C$; (b) $\left|\zeta + \frac{3}{8}\right| t^{2/3} < C$

DP: long-time asymptotics

Sector (i): $u(x,t)$ looks like a finite train of solitons

Sector (ii): $u(x,t)$ looks like a slowly decaying modulated oscillation

$$u(x,t) = \frac{c_1}{\sqrt{t}} \cdot \sin\left(c_2 t + c_3 \log t + c_4\right) + \mathrm{o}\left(\frac{1}{\sqrt{t}}\right)$$

- $c_j = c_j(\zeta; \text{scatt.data})$, $\zeta = x/t$

Sector (iii): $u(x,t) \sim$ the sum of two decaying modulated oscillations

Sector (iv): $u(x,t)$ is fast decaying

- transitions: in terms of solutions of Painlevé II equation

$$v''(s) = s v(s) + 2 v^3(s)$$

OV ($\varkappa = 1$) on $(-\infty, \infty)$ with decaying initial data

- $u_{txx} - 3u_x + 3u_x u_{xx} + u u_{xxx} = 0$
- $u(x, 0) = u_0(x)$

Let $m(x, t) := -u_{xx}(x, t)$.
Assumptions:

- $u_0(x) \to 0$ as $|x| \to \infty$
- $m(x, 0) + 1 > 0$ for all $x \in \mathbb{R}$ (then $m(x, t) + 1 > 0$ for all x, t)

Long time behavior of $u(x, t)$

- $x > 0$: fast decay (no smooth solitons)
- $x < 0$: decaying (as $t^{-1/2}$) modulated oscillations
- transition: via fast decay

OV: long time asymptotics

For $x < 0$:

$$u(x,t) = \frac{c_1}{\sqrt{t}} \cdot \sin\left(c_2 t + c_3 \log t + c_4\right) + \mathrm{o}\Big(\frac{1}{\sqrt{t}}\Big)$$

where $c_j = c_j(\xi)$ with $\xi = \sqrt{\frac{t}{|x|}}$:

- $c_1 = -2^{\frac{3}{2}} 3^{\frac{1}{4}} \sqrt{\frac{h}{\xi^3}} \sin\left(\frac{\arg r(\xi) - \arg r(-\xi)}{2} - \frac{2\pi}{3}\right)$
- $c_2 = \frac{2\sqrt{3}}{\xi}$
- $c_3 = h$
- $c_4 = h \log \frac{8\sqrt{3}}{\xi} + \frac{\arg r(\xi) + \arg r(-\xi)}{2} + \arg\Gamma(-\mathrm{i}h) + \frac{\pi}{4} +$
  $\frac{3\xi}{\pi} \int_\xi^\infty \frac{\log\left(1 - |r(s)|^2\right)}{s^2}\, \mathrm{d}s + \frac{1}{2\pi}\left(\int_{-\infty}^{-\xi} + \int_\xi^\infty\right) \frac{\log(1-|r(s)|^2)(2s+\xi)}{s^2+s\xi+\xi^2}\, \mathrm{d}s +$
  $\frac{1}{\pi}\left(\int_{-\infty}^{-\xi} + \int_\xi^\infty\right) \log|\xi - s|\, \mathrm{d}\log(1 - |r(s)|^2)$

with $h = h(\xi) = -\frac{1}{2\pi}\log(1 - |r(\xi)|^2)$.

- $r(\xi)$ is the reflection coefficient corresponding to $u_0(x)$ (spatial Lax equation)
- transition: $\xi \to \infty$ and $h(\xi) \to 0$ as $x/t \uparrow 0$

Method

Lax pair:

$$\Psi_x - \Psi_{xxx} = -\eta(m+1)\Psi \text{ (DP)} \qquad -\Psi_{xxx} = -\eta(m+1)\Psi \text{ (OV)}$$

$$\Psi_t = \frac{1}{\eta}\Psi_{xx} - u\Psi_x + \left(u_x - \frac{c}{\eta^\nu}\right)\Psi$$

DP: $m = u - u_{xx}$, $\nu = 1$ OV: $m = -u_{xx}$, $\nu = 0$

Inverse scattering transform method in the RHP form

- using
  - the Lax pair associated to the DP (OV) equation
- construct
  - a multiplicative matrix Riemann-Hilbert problem (RHP)
- obtain
  - a representation of the solution $u(x,t)$ of the DP (OV) equation in terms of the solution $\mu(x,t;\cdot)$ of the associated RHP
- obtain
  - the long-time asymptotics of $u(x,t)$ via the Deift-Zhou nonlinear steepest descent method.

Specific features of RHP for DP and OV

- In order to control analytic properties of eigenfunctions w.r.t. the spectral parameter: two versions of the Lax pair:
  - for large η
  - for η near 0
- Solution $u(x, t)$ of the DP (OV) equation: from the evaluation of the solution of the RH problem for η near 0
- Dependence on u of the exponential factor in the RH problem: requires introducing auxiliary scale, which leads to implicit (parametric) formulas for $u(x, t)$ (even for pure soliton solutions)

Martix (3×3) form of the Lax pair

$$\Phi_x = U\Phi, \qquad \Phi_t = V\Phi$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta q^3 & 1 & 0 \end{pmatrix} \quad V = \begin{pmatrix} u_x - \frac{c}{\eta} & -u & \frac{1}{\eta} \\ u+1 & -\frac{c-1}{\eta} & -u \\ u_x - \eta u q^3 & 1 & -u_x - \frac{c-1}{\eta} \end{pmatrix} \quad \text{(DP)}$$

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta q^3 & 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} u_x - c & -u & \frac{1}{\eta} \\ 1 & -c & -u \\ -\eta u q^3 & 1 & -u_x - c \end{pmatrix} \quad \text{(OV)}$$

with $q = \sqrt[3]{m+1}$

To control large-η behavior of eigenfunctions, diagonalize $U_\infty = U|_{|x|=\infty}$ and $V_\infty = V|_{|x|=\infty}$:

$$P^{-1}(\eta)U_\infty(\eta)P(\eta) = \Lambda(\eta) \equiv \operatorname{diag}\{\lambda_1(\eta), \lambda_2(\eta), \lambda_3(\eta)\}$$

DP: $\lambda_j^3 - \lambda_j = \eta$ OV: $\lambda_j^3 = \eta$

This dictates introducing a new spectral parameter k, differently for DP and for OV:

- DP: $\eta = \frac{1}{3\sqrt{3}}\left(k^3 + k^{-3}\right)$; $\lambda_j(k) = \frac{1}{\sqrt{3}}\left(\omega^j k + \omega^{-j} k^{-1}\right)$
- OV: $\eta = k^3$; $\lambda_j(k) = \omega^j k$ $(\omega = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}})$

Eigenfunctions for large k, I

Introducing $D = \mathrm{diag}\{q^{-1}, 1, q\}$ and $\hat{\Phi} = P^{-1}D^{-1}\Phi$ gives the Lax pair in the form

$$\hat{\Phi}_x - q\Lambda(k)\hat{\Phi} = \hat{U}\hat{\Phi},$$

$$\hat{\Phi}_t + (uq\Lambda(k) - A(k))\hat{\Phi} = \hat{V}\hat{\Phi}$$

($A(k) = P^{-1}(k)V_\infty(k)P(k)$ is also diagonal), where $\hat{U}$, $\hat{V}$ are $o(1)$ as $|x| \to \infty$ and $O(1)$ as $k \to \infty$ (moreover, $\mathrm{diag}(U)$, $\mathrm{diag}(V)$ are $O(1/k)$). This suggests introducing a diagonal Q solving the system

$$Q_x = q\Lambda(k), \qquad Q_t = -uq\Lambda(k) + A(k)$$

by $Q = y(x,t)\Lambda(k) + tA(k)$ with $y(x,t) = x - \int_x^\infty (q(\xi,t) - 1)\,\mathrm{d}\xi$, and the system of integral equations for entries of $M := \hat{\Phi}\mathrm{e}^{-Q}$:

$$M_{jl}(x,t,k) = E_{jl} + \int_{\infty_{j,l}}^{x} \mathrm{e}^{-\lambda_j(k)\int_x^\xi q(\zeta,t)\,\mathrm{d}\zeta} \left[(\hat{U}M)_{jl}(\xi,t,k)\right] \mathrm{e}^{\lambda_l(k)\int_x^\xi q(\zeta,t)\,\mathrm{d}\zeta}\,\mathrm{d}\xi,$$

where E is 3×3 identity matrix and

$$\infty_{jl} = \begin{cases} +\infty, & \text{if } \mathrm{Re}\,\lambda_j(k) \geq \mathrm{Re}\,\lambda_l(k), \\ -\infty, & \text{if } \mathrm{Re}\,\lambda_j(k) < \mathrm{Re}\,\lambda_l(k). \end{cases}$$

Eigenfunctions for large k, II

Propositon

Assume that u solves IVP for DP (OV). Then

- M, as function of k, is piecewise meromorphic w.r.t. $\Sigma = \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$
- $M(x,t,k) \to E$ as $k \to \infty$
- $M_+(x,t,k) = M_-(x,t,k)\mathrm{e}^{Q(x,t,k)} S_0(k) \mathrm{e}^{-Q(x,t,k)}$ for $k \in \Sigma$

Here $S_0(k)$ has only 2×2 nontrivial blocks:

$$S_0(k) = \begin{pmatrix} 1 & \overline{r(k)} & 0 \\ -r(k) & 1 - |r(k)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } k \in \mathbb{R}$$

whereas the structure of S_k on $\omega\mathbb{R}$ and $\omega^2\mathbb{R}$ follows from the symmetry $S_0(k\omega) = C^{-1} S_0(k) C$ with $C = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

Eigenfunctions for k near $\mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$ (DP) or near 0 (OV)

The behavior of M at these points is controlled by another version of the Lax pair. Introducing $\hat{\Phi}^{(0)} = P^{-1}\Phi$ leads to the Lax pair in the form

$$\hat{\Phi}^{(0)}_x - \Lambda(k)\hat{\Phi}^{(0)} = \hat{U}^{(0)}\hat{\Phi}^{(0)},$$
$$\hat{\Phi}^{(0)}_t - A(k)\hat{\Phi}^{(0)} = \hat{V}^{(0)}\hat{\Phi}^{(0)},$$

where $\hat{U}^{(0)}(x,t,k^*) \equiv 0$ with $k^* = \mathrm{e}^{\frac{\mathrm{i}\pi}{6}}$ (DP) or $k^* = 0$ (OV). This leads to the representation

$$M(x,t,k) = P^{-1}(k)D^{-1}(x,t)P(k)M^{(0)}(x,t,k)\mathrm{e}^{(x-y(x,t))\Lambda(k)}$$

with $P = \begin{pmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}$, where $M^{(0)} = \hat{\Phi}^{(0)}\mathrm{e}^{-x\Lambda - tA}$ is determined by similar integral equations; particularly, $M^{(0)}(\cdot,\cdot,k^*) \equiv E$. Moreover, in the case of OV,

$$M^{(0)}(x,t,k) = E - \frac{1}{3}u_x \begin{pmatrix} \omega & \omega & \omega \\ \omega^2 & \omega^2 & \omega^2 \\ 1 & 1 & 1 \end{pmatrix} k + O(k^2), \quad k \to 0.$$

Observations:

- the dependence of the jump matrix on (x,t): implicit (Q involves $u(x,t)$)

$$S(x,t;k) = \mathrm{e}^{Q(x,t,k)} S_0(k) \mathrm{e}^{-Q(x,t,k)}, \quad Q = y(x,t)\Lambda(k) + tA(k)$$

  with $y(x,t) = x - \int_x^\infty (q(\xi,t) - 1)\,\mathrm{d}\xi$ and

$$A(k) = \begin{cases} \sqrt{3}(k^3 + k^{-3})^{-1} E + \Lambda^{-1}(k), & \text{(DP)} \\ \Lambda^{-1}(k), & \text{(OV)} \end{cases}$$

  but becomes explicit when switching to (y,t): $Q = y\Lambda(k) + tA(k)$

- a well-controlled behavior of M at $k = k^*$ allows extracting $u(x,t)$ from $M(x,t,k)$

Solution of DP and OV via RH problem

Vector Riemann-Hilbert problem

Given $S_0(k)$, $k \in \Sigma$, find piecewise meromorphic vector $\mu(y,t,k)$ (1×3) s.t.

- $\mu_+(y,t,k) = \mu_-(y,t,k)S(y,t,k)$, $k \in \Sigma$, with
$$S(y,t,k) = \mathrm{e}^{y\Lambda(k)+tA(k)}S_0(k)\mathrm{e}^{-y\Lambda(k)-tA(k)}$$
- $\mu(y,t,k) = \begin{pmatrix}1 & 1 & 1\end{pmatrix} + o(1)$ as $k \to \infty$
- residue conditions (if any)

Then $u(x,t)$ can be obtained, in a parametric form, by $u(x,t) = \hat{u}(y(x,t),t)$, where

1. $\hat{u}(y,t) = \frac{\partial x(y,t)}{\partial t}$
2. $x(y,t)$ is given by
   - for DP: $x(y,t) = y + \log \frac{\mu_{j+1}}{\mu_j}(y,t,\mathrm{e}^{\frac{\mathrm{i}\pi}{6}})$, $j = 1$ or 2
   - for OV: $x(y,t) = y + \lim_{k\to 0}\left(\frac{\mu_3(y,t,k)}{\mu_3(y,t,0)} - 1\right)\frac{1}{k}$

Asymptotics: exponentials in jump matrix

Exponential factor in the jump matrix for RH problem dictates the contour deformations: let $\zeta = \dfrac{y}{t}$; then for $k \in \mathbb{R}$

- DP: $S_{12}(y,t,k) = \bar{r}(k)\mathrm{e}^{-3\mathrm{i}t\Theta(\frac{2}{3}\zeta,\tilde{k}(k))}$ with

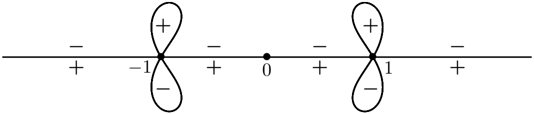
$$\Theta(\zeta,\tilde{k}) = \zeta k - \frac{2\tilde{k}}{1+4\tilde{k}^2} \text{ (as for CH)}, \quad \tilde{k}(k) = \frac{1}{2}\left(\frac{1}{k} - k\right)$$

- OV: $S_{12}(y,t,k) = \bar{r}(k)\mathrm{e}^{-2\mathrm{i}t\Theta(\zeta,k)}$ with

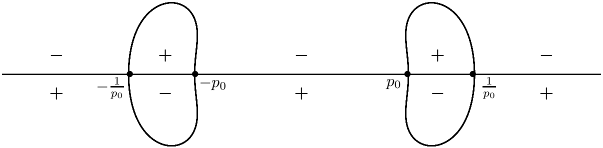
$$\Theta(\zeta,k) = -\frac{\sqrt{3}}{2}\left(\zeta k - \frac{1}{k}\right) \text{ (as for sw-CH)}$$

Asymptotics: signature table for DP, I

Sign of $\operatorname{Im}\Theta(\zeta,k)$ for various ζ; k near $\mathbb{R}$

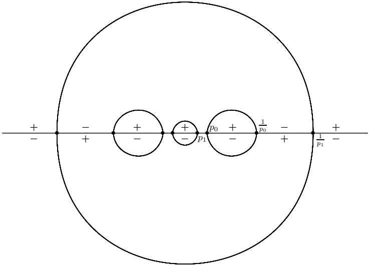


$\zeta = 3$

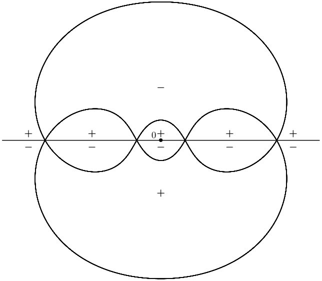


$0 < \zeta < 3$

Asymptotics: signature table for DP, II

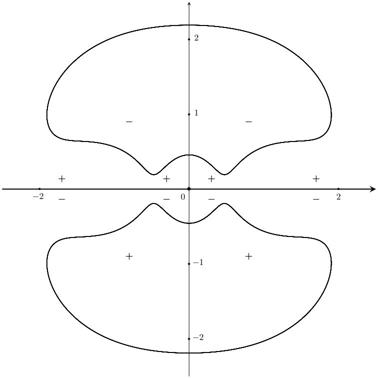


$$-\frac{3}{8} < \zeta < 0$$

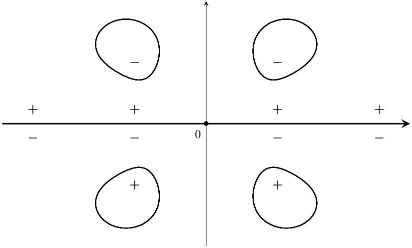


$$\zeta = -\frac{3}{8}$$

Asymptotics: signature table for DP, III

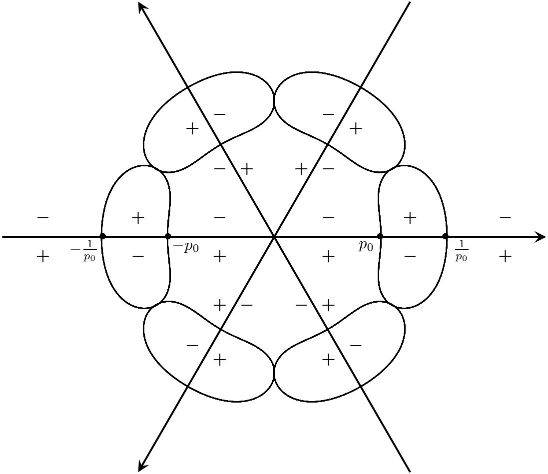


$$-1 < \zeta < -\frac{3}{8}$$

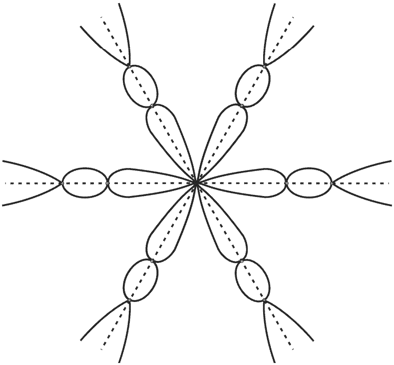


$$-1 < \zeta < -\frac{3}{8}$$

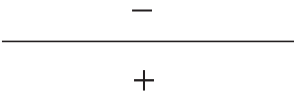
Sign of $\operatorname{Im}\Theta(\zeta,k)$ for k near each part (line) of Σ, $0<\zeta<3$



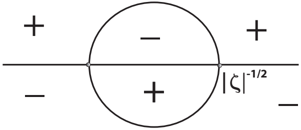
Deformation of RH problem for DP, $0 < \zeta < 3$



Signature table and contour deformation for OV

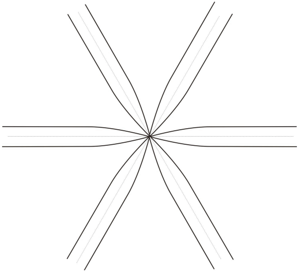


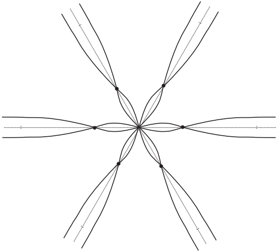
$\zeta > 0$



$\zeta < 0$

Contour deformations:





Solitons

Introducing residue conditions in the RH problem

$$\mathrm{Res}_{k=k_n} \mu_l(y,t,k) = \mu_j(y,t,k_n) v_n^{jl} \mathrm{e}^{y(\lambda_j(k)-\lambda_l(k))+t(A_j(k)-A_l(k))}$$

leads to soliton solutions.

- DP: $\mathrm{e}^{\cdots} = \mathrm{e}^{g\left(y - \frac{3t}{1-g^2}\right)}$, $g = \lambda_j(k) - \lambda_l(k)$.
  With g real and $|g| < 1$, one obtains smooth solitons in parametric form, with velocities > 3 (similarly to CH)
- OV: no smooth, real solitons (conjecture).
  But: forcing residue conditions and requiring the solution to be real and bounded leads to solutions which are smooth in the (y,t) scale but multivalued in the original (x,t) variables.

One-soliton: due to symmetries, residue conditions are at $k = \rho \mathrm{e}^{\frac{\mathrm{i}\pi}{6} + \frac{\mathrm{i}\pi m}{3}}$, $m = 0, \ldots, 5$ with $\rho > 0$:

$$\mathrm{Res}_{k=-\mathrm{i}\rho}\, \mu_1(y,t,k) = \mu_2(y,t,\mathrm{i}\rho)\gamma \mathrm{e}^{-\sqrt{3}\rho y - \frac{\sqrt{3}}{\rho} t}$$

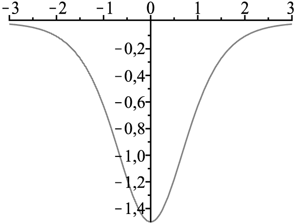
with $\gamma \equiv |\gamma| \mathrm{e}^{\frac{\mathrm{i}\pi}{3}}$ (residue conditions at other points follow by symmetries).

1-loop soliton: $u(x,t) = \hat{u}(y(x,t),t)$

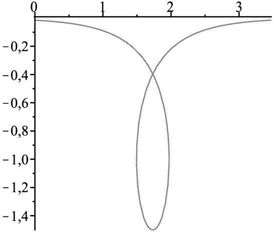
- $x(y,t) = y + \frac{2\sqrt{3}}{\rho} \frac{\hat{e}}{1+\hat{e}}$
- $\hat{u}(y,t) = -\frac{6}{\rho^2} \frac{\hat{e}}{(1+\hat{e})^2}$

where $\hat{e}(y,t) = \mathrm{e}^{-\sqrt{3}\rho(y + \frac{t}{\rho^2} + y_0)}$, $y_0 = -\frac{1}{\sqrt{3}\rho} \log \frac{|\gamma|}{2\sqrt{3}\rho}$.

OV: loop solitons, II



Soliton in (y, t)



Soliton in (x, t)

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